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# Simultaneously maximal radial cluster sets<sup>☆</sup>

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## Abstract

In this paper, we show that for a wide class of operators  $T$ —including infinite order differential operators, and multiplication and composition operators—acting on the space  $H(\mathbb{D})$  of holomorphic functions in the unit disk  $\mathbb{D}$ , we have most functions  $f \in H(\mathbb{D})$  which enjoy the property that  $Tf$  has maximal radial cluster set at any boundary point.

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## 1. Introduction and notation

Throughout this paper  $\mathbb{Z}$  will stand for the set of all integers,  $\mathbb{N}$  is the set of positive integers,  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ ,  $\mathbb{C}$  is the complex plane and  $\widehat{\mathbb{C}}$  is its one-point compactification  $\mathbb{C} \cup \{\infty\}$ . As usual,  $B(a, r)$  will denote the euclidean open ball centered at the point  $a \in \mathbb{C}$

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with radius  $r > 0$ , and  $\mathbb{D} = B(0, 1)$  is the unit open disk and  $\mathbb{T}$  the unit circle. If  $A \subset \mathbb{C}$  then  $\bar{A}$  represents its closure in  $\mathbb{C}$ . If  $G$  is a domain ( $:=$  connected, nonempty open subset of  $\mathbb{C}$ ) then  $\partial G$  ( $\partial_\infty G$ , resp.) will stand for its boundary in  $\mathbb{C}$  (in  $\widehat{\mathbb{C}}$ , resp.), while  $H(G)$  denotes the Fréchet space of holomorphic functions on  $G$ , endowed with the topology of the local uniform convergence in  $G$ . In particular  $H(G)$  is a Baire space. Let  $\mathcal{K}(G)$  be the family of all compact subsets of  $G$ ,  $\mathcal{K}_1(G)$  be the family of compact subsets  $K$  of  $G$  such that no connected component of  $G \setminus K$  is relatively compact in  $G$ , and  $\mathcal{K}_2(G)$  be the family of compact subsets  $K$  of  $G$  such that  $\mathbb{C} \setminus K$  is connected. It holds that  $\mathcal{K}_2(G) \subset \mathcal{K}_1(G) \subset \mathcal{K}(G)$  and that each  $K \in \mathcal{K}(G)$  is contained in some  $L \in \mathcal{K}_1(G)$  (see [9]). A Jordan domain is a domain  $G$  in  $\mathbb{C}$  such that  $\partial_\infty G$  is a topological image of  $\mathbb{T}$ . Finally, if  $G$  is a Jordan domain in  $\mathbb{C}$  and  $\xi \in \partial_\infty G$  then a curve in  $G$  ending at  $\xi$  is a continuous mapping  $\gamma : [0, 1) \rightarrow G$  such that  $\lim_{u \rightarrow 1^-} \gamma(u) = \xi$ ; we will denote  $\tilde{\gamma} := \gamma([0, 1))$ .

The essential background on cluster sets can be found in [8,15]. Let us recall some notions. Let  $G$  be a domain of  $\mathbb{C}$ ,  $F : G \rightarrow \mathbb{C}$  be a function and  $A$  be a subset of  $G$  with an accumulation point on  $\partial G$ . The *cluster set of  $F$  along  $A$*  is the set

$$C_A(F) = \{w \in \widehat{\mathbb{C}} : \text{there exists a sequence } (z_n)_{n \geq 1} \subset A \text{ tending to} \\ \text{some point of } \partial G \text{ such that } \lim_{n \rightarrow \infty} F(z_n) = w\}.$$

Moreover, if  $t_0 \in \partial G$  and  $t_0$  is an accumulation point of  $A$ , then the *cluster set of  $F$  along  $A$  at  $t_0$*  is the set

$$C_A(F, t_0) = \{w \in \widehat{\mathbb{C}} : \text{there exists a sequence } (z_n)_{n \geq 1} \subset A \text{ tending to } t_0 \\ \text{such that } \lim_{n \rightarrow \infty} F(z_n) = w\}.$$

It is clear that both  $C_A(F)$  and  $C_A(F, t_0)$  are closed subsets of  $\widehat{\mathbb{C}}$  and that  $C_A(F)$  is the  $\widehat{\mathbb{C}}$ -closure of  $\bigcup_{t_0 \in \partial G} C_A(F, t_0)$ . If  $A = G$  then the subscript “ $A$ ” and the expression “along  $A$ ” are usually omitted. A special important case occurs when  $G = \mathbb{D}$ ,  $t_0 \in \mathbb{T}$  and  $A$  is the radius  $A := \{ut_0 : u \in [0, 1)\}$ ; then we can define the *radial cluster set* as the set  $C_\rho(F, t_0) := C_A(F) = C_A(F, t_0)$ .

It is an interesting problem to get a holomorphic function with *maximal* cluster sets, that is, equal to  $\widehat{\mathbb{C}}$ . Given a domain  $G$ , a sequence  $(a_n)_{n \geq 1} \subset G$  without limit points in  $G$  and  $(w_n)_{n \geq 1}$  a sequence in  $\mathbb{C}$ , then an interpolation theorem due to Weierstrass (see [16, Theorem 15.13]) provides a function  $f \in H(G)$  such that  $f(a_n) = w_n$  ( $n \in \mathbb{N}$ ). By choosing  $(w_n)_{n \geq 1}$  as an enumeration of the set  $\mathbb{Q}[i] := \mathbb{Q} + i\mathbb{Q}$  of rational complex numbers, a function  $f \in H(G)$  with maximal cluster set along  $(a_n)_{n \geq 1}$  is obtained. Several authors have shown (see for instance [12,14,2]) that there is a residual set ( $:=$  a set with complement of first Baire category) of functions  $f \in H(G)$  such that for each  $j \in \mathbb{Z}$ ,  $C(f^{(j)}, t) = \widehat{\mathbb{C}}$  for all  $t \in \partial G$  ( $f^{(j)}$  is the derivative of order  $j$  of  $f$  if  $j \geq 0$ ; if  $G$  is simply connected and  $j < 0$ , then  $f^{(-j)}$  denotes any fixed antiderivative of order  $-j$  of  $f$  in  $G$ ), while in [3] the first author proves that for each non-relatively compact subset  $A$  of  $G$  the set  $\{f \in H(G) : C_A(f^{(j)}) = \widehat{\mathbb{C}} \text{ for all } j \geq 0\}$  is also residual. In particular, if  $G = \mathbb{D}$  and  $t_0 \in \mathbb{T}$  is fixed, then we obtain the existence of a residual set of functions with maximal radial cluster set at one prescribed point  $t_0 \in \partial G$ . By using Baire’s theorem, a residual set of functions  $f \in H(\mathbb{D})$  can be also obtained such that  $C_\rho(f^{(j)}, t) = \widehat{\mathbb{C}}$  for all

$j \geq 0$  and all  $t$  belonging to a prescribed dense *countable* subset of  $\mathbb{T}$ , while Tenthoff [18] provides a *dense* subset  $M$  of  $H(\mathbb{D})$  such that  $C_\varrho(f^{(j)}, t) = \widehat{\mathbb{C}}$  for all  $j \geq 0$ , all  $f \in M$  and all  $t \in \mathbb{T}$ . This result can also be obtained as a consequence of [7, Theorem 5] if one takes into account that the polynomials are dense in  $H(\mathbb{D})$ . Finally, in [6], it is shown, as a special instance of [6, Theorem 2.1], that there is a *dense linear manifold* of functions with maximal radial cluster sets at any point of  $\mathbb{T}$ . In addition, it is observed in [6, Section 3] that, as a consequence of Collingwood's maximality theorem, there is a *residual* subset of functions  $f \in H(\mathbb{D})$  such that  $C_\varrho(f, t_0)$  is maximal for all  $t_0$  belonging to some residual subset of  $\mathbb{T}$  *depending on  $f$* . In view of these results, the next question arises:

Does a residual set  $M \subset H(\mathbb{D})$  exist such that  $C_\varrho(f, t) = \widehat{\mathbb{C}}$  for all functions  $f \in M$  and all boundary points  $t \in \mathbb{T}$ ?

In this paper, we are concerned with looking for operators  $T$  acting on  $H(\mathbb{D})$  for which there exists a residual set of functions whose images under  $T$  have maximal radial cluster sets at any boundary point  $t \in \mathbb{T}$ . Examples of such operators—including infinite order differential operators, multiplication operators and composition operators—are furnished. The identity operator  $T = Id$  will be one of the operators having the mentioned maximality property, so we have a comprehensive answer to the above question.

## 2. Operators generating maximal radial cluster sets

In this section, we are going to state our main result about maximality of radial cluster sets. However, a more general situation can be studied by replacing each function  $f \in H(\mathbb{D})$  by the action  $Tf$  of an operator  $T : H(\mathbb{D}) \rightarrow H(\mathbb{D})$  on  $f$ . But before this we need some preliminary definitions, both of which were similarly stated in [5], see also [4]. By an operator on  $H(G)$  we mean a continuous (not necessarily linear) selfmapping  $T : H(G) \rightarrow H(G)$ , where  $G$  is a domain of  $\mathbb{C}$ .

**Definition 2.1.** Let  $G \subset \mathbb{C}$  be a domain and  $T : H(G) \rightarrow H(G)$  an operator. We say that  $T$  is *locally stable near the boundary* if for each  $K \in \mathcal{K}(G)$  there exists a compact subset  $M \in \mathcal{K}(G)$  such that for each compact subset  $L \in \mathcal{K}_2(G)$  with  $L \subset G \setminus M$ , each function  $f \in H(G)$  and each positive number  $\varepsilon > 0$ , there exist a compact subset  $L' \in \mathcal{K}_2(G)$  with  $L' \subset G \setminus K$  and a positive number  $\delta > 0$  such that

$$[g \in H(G) \text{ and } \|f - g\|_{L'} < \delta] \text{ implies } \|Tf - Tg\|_L < \varepsilon.$$

**Definition 2.2.** Let  $G \subset \mathbb{C}$  be a domain,  $T : H(G) \rightarrow H(G)$  an operator and  $\mathcal{A} \subset H(G)$ . We say that  $T$  has *locally dense range at  $\mathcal{A}$  near the boundary* if there exists a compact subset  $S \in \mathcal{K}(G)$ , such that for each  $f \in \mathcal{A}$ , each compact subset  $L \in \mathcal{K}_2(G)$  with  $L \subset G \setminus S$  and each positive number  $\varepsilon > 0$ , there is a function  $F \in H(G)$  such that  $\|TF - f\|_L < \varepsilon$ .

Of course, if the set  $\mathcal{A}$  is contained in the range of an operator  $T$  on  $H(G)$ , then  $T$  has dense range at  $\mathcal{A}$ . In particular, if  $T$  has dense range (that is, at  $\mathcal{A} = H(G)$ ) then  $T$  has locally dense range at  $H(G)$ . Examples of these situations will be found later. We are now ready to state an instrumental, abstract result (Theorem 2.1) that might be of independent

interest. From it, our main result (Theorem 2.2) will be derived. For the sake of simplicity, we delete the sentence “near the boundary” in the notions defined in the last two definitions.

**Theorem 2.1.** *Let  $G \subset \mathbb{C}$  be a domain and  $\{A_\alpha\}_{\alpha \in I}$  and  $\{B_\beta\}_{\beta \in J}$  be two families of subsets of  $G$  satisfying the following properties:*

- (a) *For every  $\alpha \in I$ , the set  $A_\alpha$  is relatively compact in  $G$ .*
- (b) *For every  $K \in \mathcal{K}(G)$  there exist  $\alpha \in I$  and  $L \in \mathcal{K}_2(G)$  such that  $A_\alpha \subset L \subset G \setminus K$ .*
- (c) *For all  $\alpha \in I$  and all  $\beta \in J$ ,  $A_\alpha \cap B_\beta \neq \emptyset$ .*

*Let  $T : H(G) \rightarrow H(G)$  be a continuous operator satisfying the next two properties:*

- (P)  *$T$  is locally stable.*
- (Q)  *$T$  has locally dense range at the constant functions.*

*Then the set  $M := \{f \in H(G) : \overline{Tf(B_\beta)} = \mathbb{C} \text{ for all } \beta \in J\}$  is residual in  $H(G)$ .*

**Proof.** Let  $(q_k)_k$  be an enumeration of  $\mathbb{Q}[i]$  such that each number in  $\mathbb{Q}[i]$  occurs infinitely many times in the sequence. Since  $\mathbb{Q}[i]$  is dense in  $\mathbb{C}$ , the family  $\{V_k : k \in \mathbb{N}\}$  is an open basis for the topology of  $\mathbb{C}$ , where  $V_k := B(q_k, 1/k)$ . Due to (c), we can select a point  $z_{\alpha\beta} \in A_\alpha \cap B_\beta$  for every pair  $(\alpha, \beta) \in I \times J$ . Then, trivially,  $C_\alpha := \{z_{\alpha\beta} : \beta \in J\} \subset A_\alpha$  ( $\alpha \in I$ ) and  $\{z_{\alpha\beta} : \alpha \in I\} \subset B_\beta$  ( $\beta \in J$ ). Let us denote by  $\delta_a$  the point evaluation functional

$$\delta_a : f \in H(G) \mapsto f(a) \in \mathbb{C},$$

which is linear and continuous.

From the definition of  $M$ , we observe that  $M \supset M_1$ , where

$$M_1 := \bigcap_{k \in \mathbb{N}} S_k$$

and

$$S_k := \bigcup_{\alpha \in I} \bigcap_{\beta \in J} (\delta_{z_{\alpha\beta}} \circ T)^{-1}(V_k) \quad (k \in \mathbb{N}).$$

The point is the trivial inclusion  $\bigcap_{\beta \in J} \bigcup_{\alpha \in I} D_{\alpha\beta} \supset \bigcup_{\alpha \in I} \bigcap_{\beta \in J} D_{\alpha\beta}$ , which is true for any family  $\{D_{\alpha\beta} : \alpha \in I, \beta \in J\}$  of sets.

Firstly, we will prove that every  $S_k$  is an open subset of  $H(G)$ . For this, it is enough to show that for a fixed  $\alpha \in I$ , the intersection  $\bigcap_{\beta \in J} (\delta_{z_{\alpha\beta}} \circ T)^{-1}(V)$  is an open subset of

$H(G)$  for each open subset  $V \subset \mathbb{C}$ . This, in turn, is fulfilled as soon as one proves the equicontinuity of the family  $\{\delta_z \circ T : z \in C_\alpha\} (= \{\delta_{z_{\alpha\beta}} \circ T : \beta \in J\})$  (see [17]). Indeed, given  $\varepsilon > 0$  then, evidently,  $|(\delta_z \circ T)(f)| = |Tf(z)| < \varepsilon$  for all  $z \in C_\alpha$  and all functions  $f$  belonging to the 0-neighbourhood  $U := \{f \in H(G) : \|Tf\|_{\overline{C_\alpha}} < \varepsilon\}$ ; note that  $U$  is certainly a 0-neighbourhood because  $T$  is continuous and, by (a), the set  $A_\alpha$  (so  $C_\alpha$ ) is relatively compact in  $G$ ; this shows the equicontinuity of  $\{\delta_z : z \in C_\alpha\}$ . Thus, the  $S_k$ 's ( $k \in \mathbb{N}$ ) are open and  $M_1$  is a  $G_\delta$ -subset.

Consequently, since  $H(G)$  is a Baire space, it suffices to demonstrate that every fixed  $S_k$  is dense in  $H(G)$ . Fix a positive number  $\eta > 0$ , a function  $g \in H(G)$  and a compact subset  $K \in \mathcal{K}_1(G)$ . Because the family  $\{D(g, K, \eta) : g \in H(G), K \in \mathcal{K}_1(G) \text{ and } \eta > 0\}$  (where  $D(g, K, \eta) := \{h \in H(G) : \|h - g\|_K < \eta\}$ ) is an open basis for the topology of  $H(G)$ , our goal is to see that

$$S_k \cap D(g, K, \eta) \neq \emptyset \quad (1)$$

Then, fix  $k \in \mathbb{N}$ ,  $g \in H(G)$ ,  $K \in \mathcal{K}_1(G)$  and  $\eta > 0$ , and let  $M$  be the compact subset of  $G$  given by the local stability of  $T$  when applied on the compact set  $K$ . Due to (b) and the fact that  $C_\alpha \subset A_\alpha$  ( $\alpha \in I$ ) we can find an element  $\alpha \in I$  and a compact set  $L \in \mathcal{K}_2(G)$  for which  $C_\alpha \subset L \subset G \setminus (M \cup S)$ , where  $S \in \mathcal{K}(G)$  is the compact set given by (Q). On the other hand, since  $L \subset G \setminus S$ , we can find for each  $k \in \mathbb{N}$  a function  $F_k \in H(G)$  such that

$$\|TF_k - q_k\|_L < 1/2k. \quad (2)$$

Since  $L \subset G \setminus M$ , there exist from (P) a compact set  $L' \in \mathcal{K}_2(G)$  with  $L' \subset G \setminus K$  and a positive number  $\delta_k > 0$  such that

$$[h \in H(G) \text{ and } \|F_k - h\|_{L'} < \delta_k] \text{ implies } \|TF_k - Th\|_L < 1/2k. \quad (3)$$

By construction,  $L' \cap K = \emptyset$ , so we may choose open subsets  $O_1, O_2$  of  $G$  with  $O_1 \cap O_2 = \emptyset$ ,  $K \subset O_1$  and  $L' \subset O_2$ . Consider the sets  $E := L' \cup K \in \mathcal{K}_1(G)$  (this fact is crucial, and it is true because  $K \in \mathcal{K}_1(G)$  and  $L' \in \mathcal{K}_2(G)$ ) and  $O := O_1 \cup O_2$ , and the function  $F : O \rightarrow \mathbb{C}$  defined as

$$F(z) := \begin{cases} g(z) & \text{if } z \in O_1, \\ F_k(z) & \text{if } z \in O_2. \end{cases}$$

It is clear that  $F \in H(O)$  and that  $E \subset O$ . Consequently, Runge's approximation theorem [16, Chapter 13] guarantees the existence of a rational function  $h(z)$  with poles outside  $G$  (so  $h \in H(G)$ ) such that  $\|F - h\|_E < \min\{\eta, \delta_k\}$ . In particular, we obtain  $\|g - h\|_K \leq \|F - h\|_E < \eta$ , so  $h \in D(g, K, \eta)$ .

On the other hand,  $\|F_k - h\|_{L'} \leq \|F - h\|_E < \delta_k$ ; whence, by (3), we get  $\|TF_k - Th\|_L < 1/2k$ , and by (2) and the triangle inequality one obtains  $\|Th - q_k\|_L < 1/k$ . But this implies  $|Th(z_{\alpha\beta}) - q_k| < 1/k$  for all  $\beta \in J$  because  $C_\alpha \subset L$ . Therefore  $h \in \bigcap_{\beta \in J} (\delta_{z_{\alpha\beta}} \circ T)^{-1}(V_k)$ , so  $h \in S_k$ . Consequently,  $h \in D(g, K, \varepsilon) \cap S_k$  and (1) is satisfied.  $\square$

As a consequence of this theorem, we get our desired result about radial cluster sets via operators. Of course, the identity operator  $Id$  satisfies (P) and (Q), so the next statement applies to  $Id$ , so solving the original problem proposed in the Introduction.

**Theorem 2.2.** *Let  $T$  be a continuous operator on  $H(\mathbb{D})$  satisfying properties (P) and (Q) of Theorem 2.1. Then there exists a residual set of functions  $f \in H(\mathbb{D})$  such that  $C_\varrho(Tf, t) = \widehat{\mathbb{C}}$  for all  $t \in \mathbb{T}$ .*

**Proof.** Firstly, we define

$$a_{nt} := \begin{cases} \left(1 - \frac{1}{2n}\right)t & \text{if } \Im(t) \geq 0, \\ \left(1 - \frac{1}{2n+1}\right)t & \text{if } \Im(t) < 0, \end{cases}$$

where  $\Im(z)$  denotes the imaginary part of every  $z \in \mathbb{C}$ .

Choose now  $I := \mathbb{N}$ ,  $J := \mathbb{T}$ ,  $G := \mathbb{D}$ ,  $A_n := \{a_{nt} : t \in \mathbb{T}\}$  ( $n \in \mathbb{N}$ ) and  $B_t := \{a_{nt} : n \in \mathbb{N}\}$  ( $t \in \mathbb{T}$ ). Then it is easy to see that the families  $\{A_n\}_{n \in \mathbb{N}}$  and  $\{B_t\}_{t \in \mathbb{T}}$  satisfy the hypothesis of Theorem 2.1. Hence the set

$$M := \{f \in H(\mathbb{D}) : \overline{\{Tf(a_{n,t}) : n \in \mathbb{N}\}} = \mathbb{C} \text{ for all } t \in \mathbb{T}\}$$

is residual in  $H(\mathbb{D})$ . Since  $a_{n_j t} \rightarrow t$  ( $j \rightarrow \infty$ ) for all  $t \in \mathbb{T}$  and all subsequences  $(n_j)_{j \geq 1} \subset \mathbb{N}$ , the residuality of  $M$  implies the residuality of the greater set  $\{f \in H(\mathbb{D}) : C_\varrho(Tf, t) = \widehat{\mathbb{C}} \text{ for all } t \in \mathbb{T}\}$ .  $\square$

In view of this result, from now on we will use the next definition.

**Definition 2.3.** Let  $T$  be a continuous operator on  $H(\mathbb{D})$ . We say that  $T$  has the *maximal radial cluster set property* (MRCS-property) if the set

$$\mathcal{M}_\varrho(T) := \{f \in H(\mathbb{D}) : C_\varrho(Tf, t) = \widehat{\mathbb{C}} \text{ for all } t \in \mathbb{T}\}$$

is residual in  $H(\mathbb{D})$ .

In terms of this new notation, we have shown that an operator on  $H(\mathbb{D})$  satisfying conditions (P) and (Q) has the MRCS-property.

**Remark 2.3.** In Theorem 2.2 we can replace radii by the family of all rotations  $\{e^{i\theta}\gamma : \theta \in [0, 2\pi]\}$  of any fixed curve  $\gamma$  in  $\mathbb{D}$  ending at 1 and we still have the same chaotic boundary behavior under the action of an operator  $T$  satisfying (P) and (Q). In fact, Theorem 2.1 allows us to derive a version of Theorem 2.2 for Jordan domains. Namely, if  $\Omega$  is a Jordan domain then there exists a family of curves  $\{\gamma_\xi\}_{\xi \in \partial_\infty \Omega}$ —with  $\gamma_\xi$  ending at  $\xi$  for each  $\xi \in \partial_\infty \Omega$ —with the property that if  $T$  is a continuous operator on  $H(\Omega)$  satisfying (P) and (Q), then there exists a residual set of functions  $f \in H(\Omega)$  such that  $C_{\gamma_\xi}(Tf) = \widehat{\mathbb{C}}$  for all  $\xi \in \partial_\infty \Omega$ . Indeed, the Osgood–Caratheodory theorem (see [13]) provides a homeomorphism  $\varphi$  from  $\mathbb{D}$  onto the  $\widehat{\mathbb{C}}$ -closure of  $\Omega$  whose restriction on  $\mathbb{D}$  is a holomorphic isomorphism from  $\mathbb{D}$  onto  $\Omega$ . Then define  $\gamma_\xi$  for each  $\xi \in \partial_\infty \Omega$  as  $\gamma_\xi(u) = \varphi(\varphi^{-1}(\xi)u)$  ( $0 \leq u < 1$ ). The conclusion follows by applying Theorem 2.1 to  $I := \mathbb{N}$ ,  $J := \mathbb{T}$ ,  $G := \Omega$ ,  $A_n := \varphi(\{a_{nt}\}_{t \in \mathbb{T}})$  ( $n \in \mathbb{N}$ ) and  $B_t := \varphi(\{a_{nt}\}_{n \in \mathbb{N}})$  ( $t \in \mathbb{T}$ ), in a similar way as in the proof of Theorem 2.2.

Let us give a first example of operators under whose action this behavior is induced. If  $\Phi(z) = \sum_{k \geq 0} \phi_k z^k$  is an entire function of subexponential type, that is, for all numbers  $\varepsilon > 0$  there is a constant  $A = A(\varepsilon) > 0$  such that  $|\Phi(z)| \leq A e^{\varepsilon|z|}$ , or equivalently,  $\lim_{k \rightarrow \infty} (k! |\phi_k|)^{1/k} = 0$ , then  $\Phi(D) := \sum_{k \geq 0} \phi_k D^k$  defines a linear continuous operator

on  $H(\mathbb{D})$  (actually on  $H(G)$  with  $G$  any domain in  $\mathbb{C}$ ) called an infinite order differential operator, where  $D$  denotes the differential operator and  $D^0 :=$  the identity operator. Using Cauchy's estimates, it is easy to check that if  $\Phi \neq 0$  then  $\Phi(D)$  is locally stable (in fact, we can take  $M = K$  in Definition 2.1 and  $L'$  may be a compact set slightly greater than  $L$ ) and obviously its range contains the constant functions. Hence with the help of Theorem 2.2 we get the following result.

**Theorem 2.4.** *Every non-zero infinite order differential operator  $\Phi(D)$  on  $H(\mathbb{D})$  induced by an entire function  $\Phi$  of subexponential type has the MRCS-property.*

In particular, if for each  $j \geq 0$  we take  $\Phi(z) := z^j$  then we get that the differential operator of order  $j > 0$  (the identity operator if  $j = 0$ ) has this property and, since a countable intersection of residual sets is again residual, we obtain the next corollary, which generalizes the result of Tenthoff [18] stated in the first section.

**Corollary 2.5.** *The set  $\{f \in H(\mathbb{D}) : C_\varrho(f^{(j)}, t) = \widehat{\mathbb{C}} \text{ for all } j \geq 0 \text{ and all } t \in \mathbb{T}\}$  is residual in  $H(\mathbb{D})$ .*

**Remark 2.6.** Due to Fatou's theorem on radial limits (see [10]) we cannot expect to get functions in the Hardy space  $H^p$  with maximal radial cluster set at any boundary point.

Observe that the antiderivative operator of order  $j \in \mathbb{N}$  at  $a \in \mathbb{D}$ , namely  $D_a^{-j} f :=$  the unique function  $g \in H(\mathbb{D})$  such that  $D^j g = f$  and  $D^k g(a) = 0$  ( $0 \leq k < j$ ), is not locally stable, so we cannot apply Theorem 2.2. Hence, we propose the next problem:

Is the set  $\{f \in H(\mathbb{D}) : C_\varrho(f^{(j)}, t) = \widehat{\mathbb{C}} \text{ for all } j \in \mathbb{Z} \text{ and all } t \in \mathbb{T}\}$  residual in  $H(\mathbb{D})$ ?

Of course, the problem may be formulated more generally by replacing the antiderivatives by an integral operator  $T$  given by  $Tf(z) = \int_a^z \varphi(z, t)f(t) dt$  ( $f \in H(\mathbb{D})$ ,  $z \in \mathbb{D}$ ), where  $\varphi$  satisfies adequate conditions.

### 3. Further examples

It is interesting to construct operators with the MRCS-property from others which enjoy this property. In the case of algebraic operations made on an operator  $T$  with the MRCS-property, we should "control" the boundary radial behavior of the "perturbing" operator. We joint together in the next theorem the main three operations: sum, product and composition. A nice result on large linear submanifolds is established in the last part of the assertion.

**Theorem 3.1.** (a) *Let  $T, S : H(\mathbb{D}) \rightarrow H(\mathbb{D})$  be operators such that  $T$  has the MRCS-property. We have:*

- (i) *If for every  $f \in H(\mathbb{D})$  and every  $t \in \mathbb{T}$  there exists  $\lim_{r \rightarrow 1^-} (Sf)(rt) \in \mathbb{C}$ , then  $T + S$  has the MRCS-property.*

- (ii) If for every  $f \in H(\mathbb{D})$  and every  $t \in \mathbb{T}$  there exists  $\lim_{r \rightarrow 1^-} (Sf)(rt) \in \mathbb{C} \setminus \{0\}$ , then  $T \cdot S$  has the MRCS-property.
- (iii) If  $S$  is linear and onto then  $T \circ S$  has the MRCS-property.

(b) Every onto linear operator  $S$  has the MRCS-property. Moreover, there is a dense linear submanifold  $M \subset H(\mathbb{D})$  such that  $M \setminus \{0\} \subset \mathcal{M}_\varrho(S)$ .

**Proof.** (a) Parts (i) and (ii) are very easy, so their proof is omitted. As for (iii), we need to show that  $\mathcal{M}_\varrho(T \circ S)$  is residual. For this, note that  $\mathcal{M}_\varrho(T \circ S) = S^{-1}(\mathcal{M}_\varrho(T))$ . Since  $\mathcal{M}_\varrho(T)$  is residual, there are countably many dense open subsets  $W_n \subset H(\mathbb{D})$  ( $n \in \mathbb{N}$ ) with  $\mathcal{M}_\varrho(T) \supset \bigcap_{n=1}^{\infty} W_n$ . From the continuity of  $S$  it follows that each set  $S^{-1}(W_n)$  is open.

We have that  $\mathcal{M}_\varrho(T \circ S) \supset \bigcap_{n=1}^{\infty} S^{-1}(W_n)$ , so it is enough to prove that each set  $S^{-1}(W_n)$  is dense. This is true because the image under  $S$  of an open subset is again an open subset, which in turn follows from the open mapping theorem (see [17]) since  $S$  is linear and onto and  $H(\mathbb{D})$  is an  $F$ -space.

(b) That  $S$  has the MRCS-property follows from (iii) just by taking  $T = Id$ . As a consequence of [6, Theorem 2.1] (see Section 1), there is a dense linear manifold  $M_1 \subset H(\mathbb{D})$  with  $M_1 \setminus \{0\} \subset \mathcal{M}_\varrho(Id)$ . Therefore  $M \setminus \{0\} \subset \mathcal{M}_\varrho(S)$ , where  $M := S^{-1}(M_1)$ , because  $\mathcal{M}_\varrho(S) = S^{-1}(\mathcal{M}_\varrho(Id))$ . Since  $S$  is linear,  $M$  is a linear submanifold of  $H(\mathbb{D})$ . Finally,  $M$  is dense because  $M_1$  is dense and  $S$  is an open mapping.  $\square$

We next consider the multiplication operator

$$M_\psi : f \in H(\mathbb{D}) \mapsto \psi \cdot f \in H(\mathbb{D}),$$

where  $\psi \in H(\mathbb{D})$ . The set  $Z(\psi)$  of zeros of  $\psi$  plays an important role in determining whether  $M_\psi$  has the MRCS-property.

**Theorem 3.2.** If  $\psi \in H(\mathbb{D})$  and  $Z(\psi)$  is finite then the operator  $M_\psi$  has the MRCS-property.

**Proof.** Since  $\psi$  is continuous we have that it is bounded on any compact subset of  $\mathbb{D}$ , from which one easily derives that  $M_\psi$  is locally stable. According to Theorem 2.2, it is suffices to show that  $M_\psi$  has locally dense range at the constants. In order to see this, let us set  $S := Z(\psi)$ —which is finite, so compact—and take a constant  $\alpha \in \mathbb{C}$  together with a compact set  $L \in \mathcal{K}_2(\mathbb{D})$  with  $L \subset \mathbb{D} \setminus S$ , and a number  $\varepsilon > 0$ . By Runge's theorem, there is a polynomial  $F$  (so  $F \in H(\mathbb{D})$ ) such that

$$\left| F(z) - \frac{\alpha}{\psi(z)} \right| < \frac{\varepsilon}{\max_L |\psi|} \quad \text{for all } z \in L.$$

But this implies  $\|M_\psi F - \alpha\|_L < \varepsilon$ , which tells us that  $M_\psi$  has locally dense range at the constants.  $\square$



Observe that if  $Z(\psi)$  is not finite then  $M_\psi$  cannot have locally dense range at the constants (take  $L = \{a\}$  with  $a \in Z(\psi) \setminus S$ , where  $S$  is as in Definition 2.1). Therefore, it is not possible to apply Theorem 2.2. But this does not imply that  $M_\psi$  does not have the MRCS-property, as one can see from the next example.

**Example 3.1.** There is a function  $\psi \in H(\mathbb{D})$  with infinitely many zeros in  $\mathbb{D}$  such that there exists

$$L(\theta) := \lim_{r \rightarrow 1^-} \psi(re^{i\theta}) \in \partial\mathbb{D} \quad \text{for all } \theta \in [0, 2\pi]. \quad (4)$$

Then, trivially,  $\mathcal{M}_\varrho(Id) = \mathcal{M}_\varrho(M_\psi)$ , so  $\mathcal{M}_\varrho(M_\psi)$  is residual and  $M_\psi$  has the MRCS-property. According to Frostman [11] (see also [1, Theorem 1]), if  $\theta \in [0, 2\pi]$  and  $(a_k)_{k \geq 1}$  is a sequence of distinct points in  $\mathbb{D} \setminus \{0\}$  such that

$$\sum_{k=1}^{\infty} \frac{1 - |a_k|}{|e^{i\theta} - a_k|} < \infty, \quad (5)$$

then the corresponding Blaschke product

$$\psi(z) := \prod_{k=1}^{\infty} \frac{\overline{a_k}}{a_k} \frac{a_k - z}{1 - \overline{a_k}z} \quad (z \in \mathbb{D}),$$

which is in  $H(\mathbb{D})$  and has infinitely many zeros, has its radial limit  $L(\theta) \in \partial\mathbb{D}$ . Therefore, in order that (4) can be fulfilled, it is enough to find a sequence  $a_k$  as before such that (5) holds for all  $\theta \in [0, 2\pi]$ . For this, choose  $a_k := \frac{1}{2}(1 + e^{i/k^2})$  ( $k \in \mathbb{N}$ ). This is a “good” sequence if one takes into account that after a computation of two finite Taylor expansions it is seen that  $\lim_{t \rightarrow 0} \frac{1 - |\frac{1}{2}(1 - e^{it})|}{t^{2/8}} = 1 = \lim_{t \rightarrow 0} \frac{|1 - e^{it}|}{t}$ . The last approximation covers the “worst” case  $\theta = 0$ . As for  $\theta \neq 0$ , the elementary geometrical inequality  $|e^{i\theta} - \frac{1}{2}(1 + e^{i/k^2})| > |\sin \frac{\theta}{2}|$  for all  $k \geq k_0(\theta) \in \mathbb{N}$  (use that  $(1/2)(1 + e^{i/k^2})$  tends to 1) solves the problem.

Note that the zeros of  $\psi$  in the last example have tangential approximation to the boundary. Hence, the following natural question arises:

Is there any function  $\psi$  with infinitely many zeros on a prescribed radius such that  $M_\psi$  has the MRCS-property?

Now we deal with composition and superposition operators. Given  $\varphi \in H(\mathbb{D}, \mathbb{D}) := \{g \in H(\mathbb{D}) : g(\mathbb{D}) \subset \mathbb{D}\}$  we define the (right) composition operator  $C_\varphi$  as

$$C_\varphi : f \in H(\mathbb{D}) \mapsto f \circ \varphi \in H(\mathbb{D}).$$

Let us remember that an application  $\psi : X \rightarrow X$  on a topological space  $X$  is said to be *proper* when the preimage of every compact is also compact. With this definition we can state suitably our result about composition operators.

**Theorem 3.3.** *If  $\varphi \in H(\mathbb{D})$  is proper, then  $C_\varphi$  has the MRCS-property.*

**Proof.** The result follows immediately from Theorem 2.2 because if  $\varphi$  is proper, then it is easy to check that  $C_\varphi$  is locally stable, and it is obvious that all constants are in the range of  $C_\varphi$ .  $\square$

In particular, the rotation operator  $R_\alpha : f(z) \in H(\mathbb{D}) \mapsto f(e^{i\alpha}z) \in H(\mathbb{D})$  and in general the composition operator generated by an automorphism of  $\mathbb{D}$  have the MRCS-property.

Finally, we consider the superposition (or left composition) operators. If  $\varphi$  is an entire function, the superposition operator  $L_\varphi : H(\mathbb{D}) \rightarrow H(\mathbb{D})$  is defined as

$$L_\varphi(f) = \varphi \circ f.$$

In this case we only have to suppose that  $\varphi$  is non-constant to get a complete characterization, which ends this paper.

**Theorem 3.4.** *Let  $\varphi$  be an entire function. Then the superposition operator  $L_\varphi$  generated by  $\varphi$  has the MRCS-property if and only if  $\varphi$  is non-constant.*

**Proof.** Using the little Picard theorem (see [16]) it is easy to check that  $L_\varphi$  has locally dense range at the constants. Moreover,  $L_\varphi$  is locally stable since it is continuous on  $H(\mathbb{D})$ . Hence by Theorem 2.2 the operator  $L_\varphi$  has the MRCS-property. On the other hand, it is trivial that if  $\varphi$  is constant then all cluster sets are constant, so  $L_\varphi$  does not have the MRCS-property.  $\square$

### Note added in Proof

While this paper was in press, we realized that, after adequate reasonings, an affirmative answer to the question posed in the Introduction can be derived from the results of Section 4 in the paper by S. Kierst and E. Szpilrajn [Sur certains singularités des fonctions analytiques uniformes, Fund. Math. 21 (1933) 267–294]. Nevertheless, our findings in the present paper, which include the introduction of a large class of operators  $T$ , surpass largely our first objective, that was merely the case  $T = Id$ .

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